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Axisymmetric stagnation flow obliquely impinging on a circular cylinder

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Abstract

Laminar stagnation flow, axisymmetrically yet obliquely impinging on the generators of a circular cylinder, is formulated as an exact solution of the Navier–Stokes equations. The outer stream is composed of a rotational axial flow superposed onto irrotational radial stagnation flow normal to the cylinder. The relative importance of these two flows is measured by a parameter γ. The viscous problem is reduced to a coupled pair of ordinary differential equations governed by a Reynolds number R introduced by Wang (1974). Two-term asymptotic formulae valid for large R are derived for the wall shear stress and for the position and slope of streamline attachment. These results agree well with exact numerical calculations for $R > 30$. In checking the consistency of our solution in the planar limit $R \to \infty$ we uncover and correct an error in the work of Dorrepaal (2000) for the position of viscous streamline attachment.

1. Introduction

Stagnation-point flows are ubiquitous in the sense that they appear in virtually all flow fields of engineering and scientific interest. In some situations flow is stagnated by a solid wall, while in others a free stagnation point or line exists interior to the fluid domain. Stagnation flows may be characterized as inviscid or viscous, steady or unsteady, two-dimensional or three-dimensional, symmetric or asymmetric, normal or oblique, homogeneous or two-fluid, and forward or reverse.

We continue our discussion of oblique stagnation-point flows begun in an earlier submission to this journal [1] by presenting a new exact solution for viscous axisymmetric flow stagnating obliquely on a circular cylinder. A schematic of this flow is given in Fig. 1. This study, which builds on the work of Wang [2], provides the three-dimensional counterpart to oblique planar stagnation flow, first formulated and solved by Stuart [3]. Unaware of Stuart's short note, this problem was independently rediscovered by Tamada [4] and again by Dorrepaal [5], each presenting additional results concerning the position and angle of the attaching streamline. Dorrepaal [6] reconsidered the problem to correct an error in his earlier paper. Liu [7] and Tilley and Weidman [1] studied the response of one fluid in the lower-half plane driven by an oblique stagnation flow of a second fluid in the upper-half plane. Wang [2] was the first to investigate the stagnation flow normally directed to the surface of a circular cylinder, and since then a number of variations that take into account unsteady flow effects, cylinder translation and rotation, and wall transpiration have appeared in the literature; see, for example, Gorla [19,20] and Cunning et al. [8]. As pointed out by Peregrine [9], the results of the flow studied here may help to better understand the local behavior of certain viscous axisymmetric splash patterns. Also, when heat transfer is taken into account, these axisymmetric stagnation flows may be relevant to the cooling of extruded tubes and rods using radially-inward directed fan or conical liquid jets.

In the two-dimensional counterpart to the present investigation, Stuart [3], Tamada [4] and Dorrepaal [5] each realized that an outer planar oblique stagnation flow could be constructed by superposing a tangential flow of uniform shear onto planar irrotational stagnation-point flow. It is then relatively straightforward to derive the equations governing the viscous problem, comprised of a linear equation for the cross flow coupled to the nonlinear Hiemenz [10] equation governing the normal planar

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stagnation-point flow of a viscous fluid. Following this strategy, an outer oblique radial stagnation flow impinging on a cylinder is constructed by superposition of an appropriate axial shear flow onto inviscid radial stagnation flow impinging normal to a cylinder.

The problem formulation and asymptotic analysis is given in Section 2. The governing equations depend only on the Reynolds number R and in Section 3 two-term large-R asymptotic solutions are found. Exact numerical integrations are compared with asymptotic results in Section 4 and concluding remarks are given in Section 5. An appendix is provided to correct some errors in the formulae of Dorrepaal [5,6] for the position of viscous streamline attachment of planar oblique stagnation-point flow discovered during the course of the present study.

2. Problem formulation

Dimensional variables are denoted by an asterisk and nondimensional variables are asterisk free. A sketch of an external oblique flow on a cylinder of radius a showing the cylindrical coordinate directions (r^*, z^*) is provided in Fig. 1.

For axisymmetric flow in the absence of swirl, we take (u^*, w^*) as radial and axial velocities derivable from a Stokes streamfunction ψ^*. Coordinates are nondimensionalized with a, velocities with βa, the streamfunction with βa^3, and the pressure with $\rho\nu\beta$, where ρ is the fluid density and β characterizes the strain rate of the normal stagnation flow. In dimensionless variables, the component velocities are calculated from the streamfunction using the relations

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \qquad w = \frac{1}{r}\frac{\partial\psi}{\partial r}.$$

We seek permissible external axial flows $w(r)$ that, when superposed on inviscid radial stagnation flow, form a stream impinging obliquely on a circular cylinder. The streamfunction $\psi = \Psi(r)$ for this flow must satisfy the axisymmetric, swirl-free Navier–Stokes equations. Reference to Goldstein [11] shows that $\Psi(r)$ is therefore governed by the Stokes equation

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)^2 \Psi = 0$$

which yields

$$r^4\Psi'''' - 2r^3\Psi''' + 3r^2\Psi'' - 3r\Psi' = 0. \tag{2.1}$$

The four independent solutions of this equidimensional equation

$$\Psi(r) = \{r^4,\ r^2,\ r^2\ln r,\ const.\} \tag{2.2}$$

give rise to the general z-independent axial shear flow

$$w(r) = Ar^2 + B\ln r + C \tag{2.3}$$

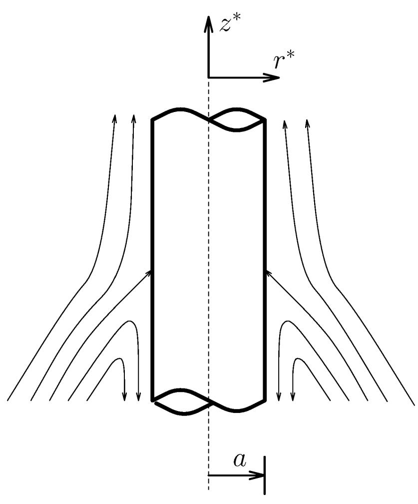

Fig. 1. Schematic of an oblique stagnation flow on a cylinder showing the dimensional cylindrical coordinates.

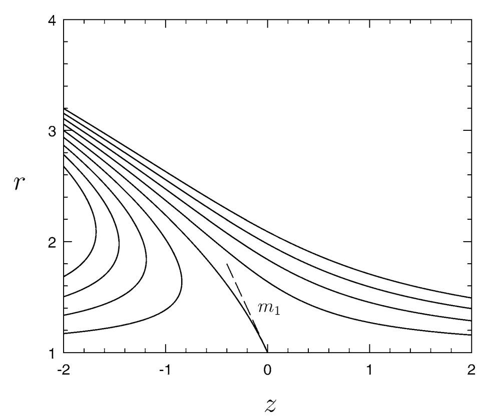

Fig. 2. Oblique streamline pattern ψ_1 computed using (2.6a) for $\gamma = 1$; ψ varies from -2.0 to 2.0 in increments of 0.5 and dashed line depict the slope m_1 of the dividing streamline at attachment.

in which A, B and C are arbitrary constants. Two distinct flows satisfying the no-slip boundary condition at $r = 1$ are obtained by choosing $A = -1/2$, $C = 1/2$, $B = 0$, on one hand, and $A = C = 0$, $B = 1$ on the other, yielding

$$w_1(r) = \frac{1}{2}\left(r^2 - 1\right), \qquad w_2(r) = \ln r. \tag{2.4a,b}$$

Substituting $r = 1 + y$ and taking the leading behavior of these velocity distributions for $y \ll 1$ yields $w_1(y) \sim y$ and $w_2(y) \sim y$, showing that both profiles tend to uniform shear in the planar limit. We seek an outer oblique stagnation flow by superposing these axial shear flows onto the irrotational velocity field describing radial stagnation flow on a cylinder, viz.

$$u = -\frac{1}{2}\left(r - \frac{1}{r}\right), \qquad w = z. \tag{2.5}$$

The coefficients for this stagnation flow impinging normal to a circular cylinder are chosen so that in the two-dimensional limit one recovers planar stagnation-point flow of strain rate β.

The relative strengths of these flows in their superposition are parameterized by weighting an axial velocity component in (2.4) with the dimensionless strain rate $\gamma = \omega/\beta$. Note that γ provides a continuous variation from pure radial stagnation flow at $\gamma = 0$ to pure axial shear flow as $\gamma \to \infty$. This furnishes two dimensionless streamfunctions $\psi_{1,2}$, corresponding respectively to axial flows $w_{1,2}$, as follows

$$\psi_1(r,z) = \frac{1}{2}\left(r^2 - 1\right)z + \frac{\gamma}{8}\left(r^2 - 1\right)^2, \tag{2.6a}$$

$$\psi_2(r,z) = \frac{1}{2}\left(r^2 - 1\right)z + \frac{\gamma}{4}\left[2r^2 \ln r - \left(r^2 - 1\right)\right]. \tag{2.6b}$$

At first glance it appears that both streamfunctions may represent a physically relevant outer flow. However, while (2.6a) satisfies the Euler equations, (2.6b) does not; hence only ψ_1 is a viable outer flow. The pressure field associated with ψ_1 is

$$p_1(r,z) = -R\left[\frac{1}{2}\left(r - \frac{1}{r}\right)^2 + 2z^2\right] + const., \tag{2.7}$$

where $R = \beta a^2/4\nu$ is the cylinder Reynolds number and ν is the kinematic viscosity of the fluid.

A sample streamline plot of ψ_1 for $\gamma = 1.0$ is provided in Fig. 2. The attachment slope m_1 of the dividing streamline $\psi_1 = 0$ is

$$m_1 = -\frac{2}{\gamma}. \tag{2.8}$$

It is noted that the slope of the dividing streamline (DSL) for ψ_1 decreases as $1/r$ for $r \to \infty$.

The viscous problem is formulated by introducing the change of variable $\eta = r^2$ and positing nondimensional velocites in the form

$$u(\eta) = -\frac{1}{2}\frac{f(\eta)}{\eta^{1/2}}, \tag{2.9a}$$

$$w(\eta, z) = zf'(\eta) + \frac{\gamma}{2}g'(\eta). \tag{2.9b}$$

Now dropping the subscript 1, the velocity and pressure fields (2.6a) and (2.7) written in (η, z) variables take the form

$$u(\eta, z) = -\frac{1}{2}\frac{(\eta - 1)}{\eta^{1/2}}, \tag{2.10a}$$

$$w(\eta, z) = z + \frac{\gamma}{2}(\eta - 1), \tag{2.10b}$$

$$p(\eta, z) = -R\left[\frac{1}{2}\frac{(\eta - 1)^2}{\eta} + 2z^2\right] + const. \tag{2.10c}$$

Inserting (2.9a,b) into the incompressible Navier–Stokes equations yields the coupled pair of ordinary differential equations

$$(\eta f'')' + R\left(ff'' - f'^2\right) = A, \tag{2.11a}$$

$$(\eta g'')' + R\left(fg'' - f'g'\right) = B, \tag{2.11b}$$

where A and B are integration constants and a prime denotes differentiation with respect to η. The boundary conditions are impermeability and no-slip at the cylinder wall and that the flow tends to (2.10) in the far-field with allowance for a possible displacement effect; therefore

$$f(1)=f'(1)=0,\quad f'(\eta)\to 1\quad (\eta\to\infty),\tag{2.12a}$$

$$g(1)=g'(1)=0,\quad g''(\eta)\to 1\quad (\eta\to\infty).\tag{2.12b}$$

Evaluation of (2.11a) in the far-field using (2.12a) gives $A=-R$. Thus the problem for $f(\eta)$ is exactly that found by Wang [2] except that our scaling gives a Reynolds number half of his. For this value of A the pressure gradients for the viscous flow are

$$-\frac{\partial p}{\partial\eta}=R\left(\frac{ff'}{\eta}-\frac{f^2}{2\eta^2}\right)+f'',\tag{2.13a}$$

$$-\frac{\partial p}{\partial z}=2R(2z-\gamma B).\tag{2.13b}$$

Matching the axial pressure gradient given in (2.13b) with its far-field counterpart calculated from (2.10c) gives $B=0$. Hence the pressure distribution for the oblique stagnation flow found from integration of (2.13) is given by

$$p(\eta,z)=-R\left(\frac{f^2}{2\eta}+2z^2\right)+f'+const.\tag{2.14}$$

Note that the pressure is not affected by the γ-component of the axial flow in (2.9b).

The far-field behavior of $f(\eta)$ governed by (2.11a) with $A=-R$, worked out using the method of dominant balance (Bender and Orszag [18]) is

$$f(\eta)\sim\left[(\eta-1)-\eta_0\right]+C\eta^{-[3-R(1+\eta_0)]}\,\mathrm{e}^{-R\eta}+\cdots\quad (\eta\to\infty),\tag{2.15}$$

where C is an undetermined constant and η_0 is calculated for each value of R as the limit

$$\eta_0=\lim_{\eta\to\infty}\left[(\eta-1)-f(\eta)\right].\tag{2.16}$$

The linear equation governing the asymptotic behavior of $g(\eta)$ is obtained by inserting (2.15) and its derivative into (2.11b) with $B=0$. Both asymptotic behaviors for $g'(\eta)$ have the form

$$g'(\eta)\sim\left[(\eta-1)-\eta_0\right]+\frac{1}{R}+\text{E.S.T.}\quad (\eta\to\infty).\tag{2.17}$$

3. Asymptotic analysis for large *R*

It is desirable to obtain the asymptotic nature of solutions for large values of R where numerical integration becomes increasingly difficult. We proceed by introducing $\varepsilon=R^{-1/2}$ and make the change of variables

$$f(\eta)=\varepsilon F(\xi),\quad g(\eta)=\varepsilon^2 G(\xi),\quad \eta=1+\varepsilon\xi$$

to transform boundary-value problem (2.11) and (2.12) into

$$(1+\varepsilon\xi)F'''+\varepsilon F''+\left(FF''-F'^2+1\right)=0;\quad F(0)=F'(0)=0,\quad F'(\infty)\to 1,\tag{3.1a}$$

$$(1+\varepsilon\xi)G'''+\varepsilon G''+\left(FG''-F'G'\right)=0;\quad G(0)=G'(0)=0,\quad G''(\infty)\to 1.\tag{3.1b}$$

This procedure was motivated by Wang [2], but here a typographical sign error in the exponent of his Eq. (15) is corrected. Inserting the regular perturbation expansions

$$F(\xi)=F_0(\xi)+\varepsilon F_1(\xi)+\cdots,$$

$$G(\xi)=G_0(\xi)+\varepsilon G_1(\xi)+\cdots$$

into (3.1) one finds the O(1) problem

$$F_0'''+F_0F_0''-F_0'^2+1=0;\quad F_0(0)=F_0'(0)=0,\quad F_0'(\infty)\to 1,\tag{3.2a}$$

$$G_0'''+F_0G_0''-F_0'G_0'=0;\quad G_0(0)=G_0'(0)=0,\quad G_0''(\infty)\to 1,\tag{3.2b}$$

and at $O(\varepsilon)$ we have

$$F_1''' + \xi F_0''' + F_0''(1 + F_1) + F_0 F_1'' - 2F_0' F_1' = 0, \tag{3.3a}$$

$$G_1''' + \xi G_0''' + G_0''(1 + F_1) + F_0 G_1'' - F_0' G_1' - G_0' F_1' = 0, \tag{3.3b}$$

$$F_1(0) = F_1'(0) = F_1'(\infty) = 0; \qquad G_1(0) = G_1'(0) = G_1''(\infty) = 0. \tag{3.3c}$$

The leading order problem (3.2a) for $F_0(\xi)$ corresponds exactly to Hiemenz [11] stagnation flow and has the known asymptotic behavior (cf. Rosenhead [17])

$$F_0 \sim \xi - \xi_0 \qquad (\xi \to \infty), \quad \xi_0 = 0.64790 \tag{3.4}$$

and wall shear stress parameter value $F_0''(0) = 1.23259$. Integration of systems (3.2) and (3.3) provides the remaining critical values $G_0''(0) = 0.60795$, $F_1''(0) = 0.27331$ and $G_1''(0) = 1.04573$. Working back through the transformations then furnishes the two-term large-R asymptotic behaviors

$$f''(1) \sim 1.23259 R^{1/2} + 0.27331, \tag{3.5a}$$

$$g''(1) \sim 0.60795 + 1.04573 R^{-1/2}. \tag{3.5b}$$

The results in (3.5a,b) can be used to produce asymptotic formulae for important physical parameters characterizing the flow. Of particular interest is the axial shear stress τ_w^* at the cylinder wall

$$\tau_w^* = \rho\nu \left[\frac{\partial w^*}{\partial r^*}\right]_{r^*=a} = 2\rho\nu\beta \left[z f''(1) + \frac{\gamma}{2} g''(1)\right]. \tag{3.6}$$

4. Presentation of results

Numerical integration of Eqs. (2.11) satisfying boundary conditions (2.12) have been performed using standard shooting techniques over the parameter range $0.1 < R < 30$. A comparison of computed values of the wall shear stress parameters $f''(1)$ and $g''(1)$ with their asymptotic behaviors at large R is given in Fig. 3. Note that the numerical and asymptotic results for $f''(1)$ are indistinguishable in this plot for all $0.1 < R < 30$.

Streamline patterns are readily obtained by evaluation of the dimensionless streamfunction

$$\psi(\eta, z) = \frac{z}{2} f(\eta) + \frac{\gamma}{4} g(\eta). \tag{4.1}$$

Fig. 4 exhibits a typical streamline pattern calculated for $R = 20$ and $\gamma = 1$. Observe that the stagnation circle lies to the left of the origin, and recall from Fig. 2 that the origin is the stagnation circle for the outer inviscid flow. Moreover, the slope of the viscous DSL at attachment is greater than the attachment slope of the outer inviscid DSL. These local geometric features are depicted in Fig. 5 where we follow Dorrepaal [5] and define $m \equiv m_1$ as the slope of the inviscid DSL at attachment and m_s as the attachment slope of the viscous DSL; furthermore, we define z_s as the axial position of the viscous stagnation circle.

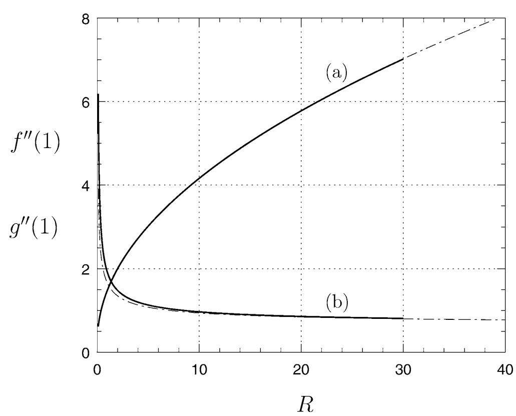

Fig. 3. Shear stress parameters (a) $f''(1)$ and (b) $g''(1)$ as a function of R showing both the numerical values (solid curves) and the two-term asymptotic approximations (dashed curves) computed using (3.5a,b).

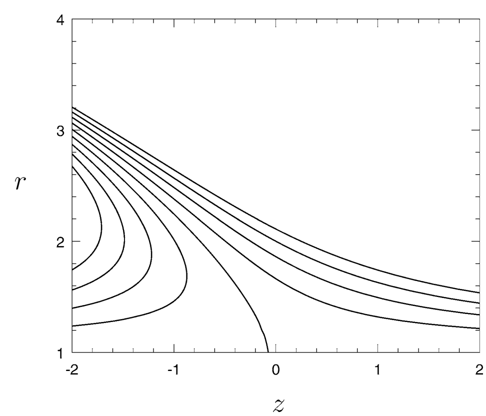

Fig. 4. Streamline pattern for $R = 20$ and $\gamma = 1$; ψ varies from -2.0 to 2.0 in increments of 0.5.

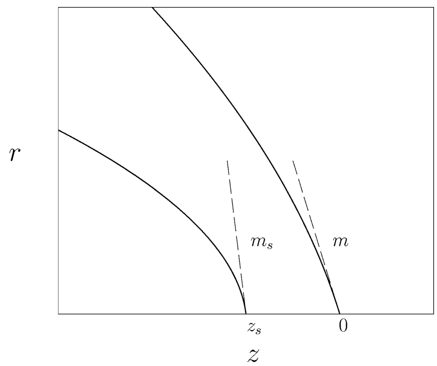

Fig. 5. Sketch depicting the geometry of inviscid and viscous dividing streamlines at attachment; m is the slope of the inviscid DSL at its intersection $z=0$ and m_s is the slope of the viscous DSL at its intersection $z=z_s$.

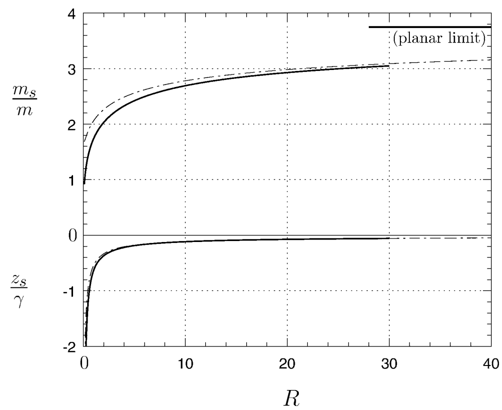

Fig. 6. Values m_s/m and z_s/γ as a function of R showing both the numerical values (solid curves) and asymptotic approximations (dashed curves) computed using (4.6) and (4.7). Also shown is the planar limit $m_s/m=3.74851$ to which the present results tend as $R\to\infty$.

Mathematical expressions for m_s and z_s are readily calculated. The dividing streamsurface $\psi=0$ intersects the cylinder at the axial position for which the shear stress is zero. This position determined from (3.6) is

$$z_s=-\frac{\gamma}{2}\frac{g''(1)}{f''(1)}. \tag{4.2}$$

The slope of the DSL at attachment is computed from the limit

$$m_s=\left(\frac{\mathrm{d}r}{\mathrm{d}z}\right)_{r\to 1}=-\lim_{\eta\to 1}\left(\frac{f'^2}{fg'-gf'}\right) \tag{4.3}$$

which is indeterminate. Four successive applications of l'Hôpital's rule finally yield

$$m_s=-\frac{3}{\gamma R}\frac{(f''(1))^2}{g''(1)}, \tag{4.4}$$

where use of (2.11a,b) has been made to eliminate $f'''(1)$ and $g'''(1)$, respectively. In the planar problem, Dorrepaal [5] has shown that the ratio $m_s/m=3.74851$ is independent of γ. While the independence on γ is also true here, the ratio m_s/m now depends on R through the relation

$$\frac{m_s}{m}=\frac{3}{2R}\frac{(f''(1))^2}{g''(1)}. \tag{4.5}$$

Two-term asymptotic behaviors for z_s and m_s/m in (4.2) and (4.5) calculated using (3.5a,b) are

$$z_s\sim-\left(0.24662R^{-1/2}+0.36952R^{-1}\right)\gamma \tag{4.6}$$

and

$$\frac{m_s}{m}\sim 3.74851-4.78540R^{-1/2}. \tag{4.7}$$

A comparison of these results with exact numerical calculations is given in Fig. 6. Note the slow convergence of m_s/m to the planar limit 3.74851 found by Dorrepaal [5]; to arrive within 1% of this limit one would need $R\sim 16,000$ in (4.7).

The dimensional shift of the viscous attachment point x_s^* from its inviscid position for planar oblique stagnation flow on a flat plate calculated from (4.6) is

$$x_s^*=\lim_{a\to\infty}(az_s)=-0.49323\frac{\nu^{1/2}\omega}{\beta^{3/2}}. \tag{4.8}$$

While this is consistent with results derived from Stuart [3] and Tamada [4], it does not agree with the value reported by Dorrepaal [6]. In our variables his result

$$x_s^*=-0.81718\frac{\nu^{1/2}\omega}{\beta^{3/2}}. \tag{4.9}$$

predicts a shift some 65% greater than (4.8) for fixed values of ω, β and ν. In the appendix we show that the error in Dorrepaal's result (4.9) can be traced to a mistake in matching the viscous solution to the outer flow.

5. Conclusion

A new stagnation flow describing the axisymmetric oblique impingement of viscous fluid on a circular cylinder has been obtained as an exact solution of the Navier Stokes equations. It is a member of a class of solutions described by Lin [12]. For practical purposes, as with all stagnation flows, the solution represents only the local behavior of motion in the vicinity of stagnation. The system is reduced to a nonlinear equation governing the radially inward stagnation flow and a coupled linear equation for the superposed axial flow component. The two-parameter family of solutions is characterized by $R = \beta a^2/4\nu$, a Reynolds number measuring the effect of cylinder curvature, and γ a parameter measuring the relative strength of the external Poiseuille flow to the radial stagnation flow. Two-term asymptotic expressions for the wall shear stress τ_w^*, the axial location z_s of the stagnation circle, and the slope m_s of the dividing streamline at attachment are shown to provide excellent approximations to the exact numerical solutions for $R > 30$. Moreover, the large radius limit of az_s provides, for the first time, the correct dimensional shift x_s^* of the viscous attachment point relative to its invisid position for planar, oblique stagnation-point flow. We note that Eq. (4.8) is consistent with results reported by Stuart [3] and Tamada [4].

Several effects that modify axisymmetric oblique stagnation flow on a cylinder readily may be accommodated within the present formulation. Perhaps the most important are cylinder translation, cylinder rotation, and uniform suction or blowing through permeable cylinder walls. Another flow likely to have some practical application is the response of a cylindrical liquid column of density ρ_1 and viscosity ν_1 to an external oblique stagnation flow of another fluid with properties ρ_2 and ν_2. This latter problem is the axisymmetric analog of the planar two-fluid oblique stagnation flow studied by Liu [7] and Tilley and Weidman [1].

Acknowledgements

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Editor's comment

Professor Dorrepaal has been consulted regarding this work and acknowledges the errors appearing in both [5,6] papers.

Appendix. The rôle of displacement thickness

Of the four studies on planar oblique stagnation flow, only Dorrepaal [5,6] gives a general formula for the shift of the attachment point of the viscous flow relative to its position given by the outer inviscid flow. Since the planar limit of our solution gives a viscous attachment point different from both of Dorrepaal's results, it is incumbent on us to elucidate clearly the discrepancies.

Historically, the problem is related to a lively controversy that took place in the Reader's Forum of the *Journal of Aero/Space Science* nearly a half century ago between Li [13,14] and Glauert [15,16] on the nature of the solution of the boundary-layer equations when the external flow is comprised of a uniform (rotational) shear superposed on a uniform (irrotational) stream. Among other things, Li claimed that there was no displacement of the rotational component of the flow while Glauert argued it must experience a displacement identical to that of the uniform stream given by the Blasius solution. Stuart's [3] note in the Reader's Forum on planar oblique stagnation flow lent support to Glauert's viewpoint since his exact solution showed that both outer components of the flow, the irrotational normal stagnation flow and a superposed uniform shear flow tangential to the plate, exhibit identical positive displacements. Two decades later Tamada [4] independently arrived at the same conclusion.

To pinpoint the perhaps subtle error in Dorrepaal's analyses, we briefly formulate the planar problem. All parameters defined in the body of the text retain their definitions and again variables with an asterisk are dimensional. In the planar problem let (x^*, y^*) be Cartesian coordinates respectively tangential and normal to a flat plate, and let (u^*, v^*) be velocities positive along the coordinate directions. Consider an outer flow consisting of a linear superposition of irrotational stagnation-point flow of strain rate β and a uniform shear of strain rate ω, viz. $u^* = \beta x^* + \omega y^*$, $v^* = -\beta y^*$. Since this flow violates the no-slip boundary condition at the wall, we will refer to it as the outer inviscid (but rotational) solution. Scaling lengths with $(\nu/\beta)^{1/2}$, velocities with $(\nu\beta)^{1/2}$, pressure with $\rho\nu\beta$, the dimensionless Navier–Stokes equations become

$$uu_x + vu_y = -p_x + u_{xx} + u_{yy}, \tag{A.1a}$$

$$uv_x + vv_y = -p_y + v_{xx} + v_{yy}. \tag{A.1b}$$

The dimensionless outer inviscid velocity field and its associated pressure distribution calculated from (A.1) are

$$u(x,y)=x+\gamma y,\qquad v(x,y)=-y,\qquad p(x,y)=-\frac{1}{2}\left(x^2+y^2\right)+const.,\tag{A.2}$$

where $\gamma=\omega/\beta$. For the viscous flow we assume

$$u(x,y)=xf'(y)+\gamma g'(y),\qquad v=-f(y).\tag{A.3}$$

Substituting (A.3) into (A.1) and eliminating the pressure yields the coupled pair of ordinary differential equations

$$\left(f'''+ff''-f'^2\right)'=0\quad\Rightarrow\quad f'''+ff''-f'^2=A,\tag{A.4a}$$

$$\left(g'''+fg''-f'g'\right)'=0\quad\Rightarrow\quad g'''+fg''-f'g'=B,\tag{A.4b}$$

where A and B are constants and a prime denotes differentiation with respect to y. The boundary conditions are that the velocity is zero at $y=0$ and that the velocity and pressure tend to the outer rotational solution (A.2) as $y\to\infty$, apart from a possible displacement of the y-coordinate. This furnishes the boundary conditions

$$f(0)=0,\qquad f'(0)=0,\qquad f'(y)\to 1\quad(y\to\infty),\tag{A.5a}$$

$$g(0)=0,\qquad g'(0)=0,\qquad g''(y)\to 1\quad(y\to\infty).\tag{A.5b}$$

Evaluation of (A.4a) in the far field using (A.5a) gives $A=-1$ which is the Hiemenz problem for normal planar stagnation-point flow. The well known asymptotic behavior derived from its solution (cf. [17]) is $f(y)\sim y-\delta$, where $\delta\approx 0.64790$. The pressure gradients for the viscous oblique stagnation-point flow are thus

$$-p_x=x-\gamma B,\tag{A.6a}$$

$$-p_y=ff'+f''.\tag{A.6b}$$

Matching the x-pressure gradient of the viscous flow (A.6a) with that derived from (A.2) requires $B=0$. Then the leading behavior of $g(y)$, readily evaluated from (A.4b) using the asymptotic result for $f(y)$, takes the form

$$g(y)\sim\frac{1}{2}(y-\delta)^2+const.\tag{A.7}$$

Solution of the system (A.4a) and (A.5a) with $A=-1$ gives the wall shear stress parameter $f''(0)=1.23259$ (cf. [17]). Solution of the system (A.4b) and (A.5b) with $B=0$ reported by Stuart [3], Tamada [4], and Dorrepaal [5,6] give, in the present scaling, $g''(0)=0.60795$. The position x_s of attachment of the dividing streamline is determined by the point of zero wall shear stress, viz.

$$\left.\frac{\partial u}{\partial y}\right|_{y=0}=x_s f''(0)+\gamma g''(0)\quad\Rightarrow\quad x_s=-0.49323\gamma.$$

To avoid any confusion amongst the various scalings used in this problem, we give the position of the attachment streamline in dimensional form

$$x_s^*=-0.49323\frac{\nu^{1/2}\omega}{\beta^{3/2}}\tag{A.8}$$

noting that it corrects both of Dorrepaal's results listed below

$$x_s^*=-1.14223\frac{\nu^{1/2}\omega}{\beta^{3/2}}\quad\text{(Dorrepaal [5])},$$

$$x_s^*=-0.81718\frac{\nu^{1/2}\omega}{\beta^{3/2}}\quad\text{(Dorrepaal [6])}.$$

We note that the asymptotic behavior of the velocity and pressure fields of the viscous solution

$$u(x,y)\sim x+\gamma(y-\delta),\qquad v(x,y)\sim-(y-\delta),$$

$$p(x,y)\sim-\frac{1}{2}\left[x^2+(y-\delta)^2\right]+const.$$

shows that the entire outer inviscid flow field impinging on the horizontal plane is shifted outward by nondimensional distance δ. This corresponds to the dimensional shift $\delta^*=\delta(\nu/\beta)^{1/2}=0.64790(\nu/\beta)^{1/2}$ known as the displacement thickness in the aerodynamics community.

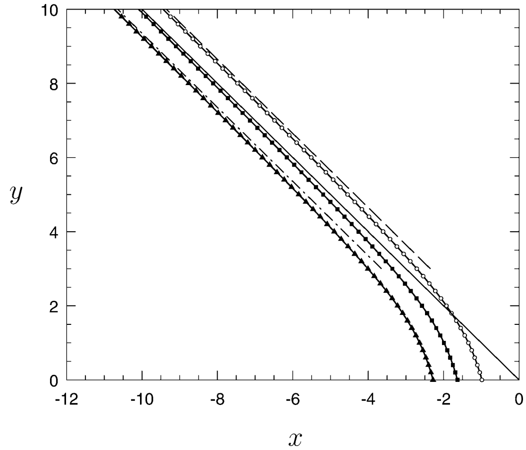

Fig. 7. Dividing streamlines for viscous planar oblique stagnation-point flow for $\gamma = 2$; open circles with the dashed line asymptote represent the equivalent results of Stuart [3] and Tamada [4] with attachment at $x = -0.98647$; solid triangles with the dash–dot–dashed line asymptote is the result of Dorrepaal [5] with attachment at $x = -2.28226$; solid squares with the solid line asymptote is the result of Dorrepaal (1986) with attachment at $x = -1.63436$; the solid line through the origin is the dividing streamline of the inviscid flow.

The three dividing streamlines obtained here and by Dorrepaal [5,6] are displayed in Fig. 7 for an outer flow inclined at 45° to the horizontal, corresponding to $\gamma = 2$. One sees that each solution is a simple horizontal shift of the other, but the important feature is that Dorrepaal's [5] result gives a negative displacement while Dorrepaal's [6] result gives no displacement. The correct solution, obtained by matching the x-pressure gradients of the viscous and outer inviscid flows, gives the physically correct positive displacement δ^* of the dividing streamline. This result is in agreement with the original findings reported by Stuart [3] and Tamada [4].

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